

Cancellation problem for projective modules over affine algebras

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1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let A be a ring of dimension d and let P be a projective A -module of rank n . We say that P is *cancellative* if $P \oplus A^m \simeq Q \oplus A^m$ for some projective A -module Q implies $P \simeq Q$.

A classical result of Bass (2.2) says that if $n > d$, then P is cancellative. It is well known that Bass' result is best possible in general (since tangent bundle of real 2-sphere is stably trivial but not trivial). However, Bass' result can be improved in some specific cases which we describe below.

Theorem 1.1 (i) *If A is an affine algebra of dimension d over an algebraically closed field, then Suslin [20] proved that every projective A -module of rank $\geq d$ is cancellative.*

(ii) *If A is an affine algebra of dimension d over an infinite perfect C_1 -field k such that $1/d! \in k$, then Suslin [19] proved that A^d is cancellative. Subsequently, Bhatwadekar ([3], Theorem 4.1) proved that every projective A -module of rank d is cancellative.*

(iii) *If A is an affine algebra of dimension d over \mathbb{Z} , then Vaserstein ([23], Corollary 18.1, Theorem 18.2) proved that A^d is cancellative. Subsequently, Mohan Kumar, Murthy and Roy ([11], Corollary 2.5) proved that every projective A -module of rank d is cancellative.*

We note that Bhatwadekar's proof [3] uses Suslin's result [19] that A^d is cancellative. Similarly, the proof of Mohan Kumar et. al. [11] uses Vaserstein's results [23]. Hence, in view of the above results, we can ask the following:

Question 1.2 *Let A be a ring of dimension d . Assume that A^d is cancellative. Is every projective A -module of rank d cancellative?*

In ([2], Example 3.11), Bhatwadekar has given an example of a smooth real affine surface A such that A^2 is cancellative, but $K_A \oplus A$ is not cancellative, where K_A is the canonical module of A . Thus, the above question has negative answer in general. We will modify the above question and prove the following result (3.5).

Theorem 1.3 *Let A be a ring of dimension d . Assume that for every finite extension R of A , R^d is cancellative. Then every projective A -module of rank d is cancellative.*

For a ring k , a finite extension of an affine k -algebra is an affine k -algebra. Hence, in (1.1), assuming the result of Suslin [19], our result gives an alternative proof of Bhatwadekar's result [3]. Similarly, assuming the result of Vaserstein [23], it gives an alternative proof of Mohan Kumar et. al. [11].

Regarding question (1.2), Bhatwadekar ([2], Proposition 3.7) proved the following interesting result: Let A be a ring of dimension 2 and let P be a projective A -module of rank 2. If $\wedge^2(P) \oplus A$ is cancellative,

then P is cancellative. In particular, if A^2 is cancellative, then every projective A -module of rank 2 with trivial determinant is cancellative. In view of this result, Bhatwadekar ([4], Question VII) asked the following question which is open for $d \geq 3$.

Question 1.4 *Let A be a ring of dimension d . Assume that A^d is cancellative. Is every projective A -module of rank d with trivial determinant cancellative?*

In [10], Mohan Kumar has given an example of a smooth affine algebra of dimension $n \geq 4$ over which there exist projective modules of rank $n - 2$ that are not cancellative. More precisely, he proved the following: let p be a prime integer and let k be any algebraically closed field. Then there exists an $f \in A = k[X_1, \dots, X_{p+2}]$ and a projective A_f -module P of rank p such that $P \oplus A_f \xrightarrow{\sim} A_f^{p+1}$ but $P \not\xrightarrow{\sim} A_f^p$, i.e. P is not cancellative.

In view of the above results, the only case remaining regarding cancellation problem is when rank $P = \dim A - 1$.

Question 1.5 *Let A be an affine algebra of dimension $n \geq 3$ over an algebraically closed field k . Let P be a projective A -module of rank $n - 1$. Is P cancellative?*

This is not known even when $n = 3$ and $P = A^2$. We prove the following result (3.7) which is analogue of (1.3) for affine algebras over $\overline{\mathbb{F}}_p$.

Theorem 1.6 *Let A be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$. Assume that if R is a finite extension of A , then R^{d-1} is cancellative. Then every projective A -module of rank $d - 1$ is cancellative.*

Let R be an affine algebra of dimension $d - 1$ over an algebraically closed field k with $1/(d - 1)! \in R$. Then Wiemers (2.12) proved that projective $R[X]$ -modules of rank $d - 1$ are cancellative, thus answering question (1.5) in affirmative in the polynomial ring case. We prove the following two results (4.1) and (6.3, 6.5) which answers question (1.5) in affirmative in some special cases.

Theorem 1.7 *Let k be an algebraically closed field with $1/d! \in k$ and let R be an affine k -algebra of dimension d . Assume that $f(T) \in R[T]$ is a monic polynomial and either*

- (i) $A = R[T, 1/f]$ or
 - (ii) $A = R[T, f_1/f, \dots, f_r/f]$, where $f, f_1, \dots, f_r \in R[T]$ is a regular sequence.
- Then A^d is cancellative.*

Theorem 1.8 *Let R be an affine algebra of dimension d over an algebraically closed field k with $1/d! \in k$. Let P be a projective $R[X, X^{-1}]$ -module of rank d . Then*

- (i) P is cancellative and
- (ii) the natural map $\text{Aut}(P) \rightarrow \text{Aut}(P/(X - 1)P)$ is surjective.

We also prove the analogue of above results for affine algebras over real closed fields (5.2, 5.3, 5.7, 5.4). Note that (ii) extends our earlier result ([7], Theorem 4.7), where it is proved for projective A -modules which are extended from R under the assumption that R is smooth.

Theorem 1.9 *Let k be a real closed field and let R be an affine k -algebra of dimension $d - 1 \geq 2$. Assume that $f(T) \in R[T]$ is a monic polynomial which does not belong to any real maximal ideal. Then the following holds:*

- (i) *If $A = R[T, 1/f]$, then every projective A -module of rank d is cancellative.*
- (ii) *If $A = R[T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a $R[T]$ -regular sequence, then every projective A -module of rank d with trivial determinant is cancellative.*
- (iii) *Further, if $R = B[X]$, then A^{d-1} is also cancellative in (i, ii).*

2 Preliminaries

Let B be a ring and let P be a projective B -module. Recall that $p \in P$ is called a *unimodular element* if there exists a $\psi \in P^* = \text{Hom}_B(P, B)$ such that $\psi(p) = 1$. We denote by $\text{Um}(P)$, the set of all unimodular elements of P . We write $O(p)$ for the ideal of B generated by $\psi(p)$, for all $\psi \in P^*$. Note that, if $p \in \text{Um}(P)$, then $O(p) = B$. For an ideal $J \subset B$, we denote by $\text{Um}^1(B \oplus P, J)$, the set of all $(a, p) \in \text{Um}(B \oplus P)$ such that $a \in 1 + J$ and by $\text{Um}(B \oplus P, J)$, the set of all $(a, p) \in \text{Um}^1(B \oplus P, J)$ such that $p \in JP$.

Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and hence $1 + \varphi_p$ is a uni-potent automorphism of P . By a *transvection*, we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either $\varphi \in \text{Um}(P^*)$ or $p \in \text{Um}(P)$. We denote by $E(P)$, the subgroup of $\text{Aut}(P)$ generated by all transvections of P . Note that $E(P)$ is a normal subgroup of $\text{Aut}(P)$.

An existence of a transvection of P pre-supposes that P has a unimodular element. Let $P = B \oplus Q$, $q \in Q, \alpha \in Q^*$. Then the automorphisms Δ_q and Γ_α of P defined by $\Delta_q(b, q') = (b, q' + bq)$ and $\Gamma_\alpha(b, q') = (b + \alpha(q'), q')$ are transvections of P . Conversely, any transvection Θ of P gives rise to a decomposition $P = B \oplus Q$ in such a way that $\Theta = \Delta_q$ or $\Theta = \Gamma_\alpha$.

For an ideal $J \subset B$, we denote by $EL^1(B \oplus P, J)$, the subgroup of $E(B \oplus P)$ generated by Δ_q and $\Gamma_{a\phi}$, where $q \in P, a \in J, \phi \in P^*$.

We begin by stating two classical results due to Serre [17] and Bass [1].

Theorem 2.1 *Let A be a ring of dimension d . Then any projective A -module of rank $> d$ has a unimodular element. In particular, if $\dim A = 1$, then any projective A -module of trivial determinant is free.*

Theorem 2.2 *Let A be a ring of dimension d and let P be a projective A -module of rank $> d$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. In particular, P is cancellative.*

The following two results are due to Lindel ([9], Theorem 2.6 and Lemma 1.1).

Theorem 2.3 *Let A be a ring of dimension d and $R = A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Let P be a projective R -module of rank $\geq \max(2, d + 1)$. Then $E(R \oplus P)$ acts transitively on $\text{Um}(R \oplus P)$. In particular, projective R -modules of rank $> d$ are cancellative.*

Lemma 2.4 *Let A be a ring and let P be a projective A -module of rank r . Then there exists $s \in A$ such that the following holds:*

- (i) P_s is free,
- (ii) there exists $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in \text{Hom}(P, A)$ such that $(\phi_i(p_j)) = \text{diagonal}(s, \dots, s)$,
- (iii) $sP \subset p_1A + \dots + p_rA$,
- (iv) the image of s in A_{red} is a non-zero-divisor and
- (v) $(0 : sA) = (0 : s^2A)$.

The following result is due to Bhatwadekar and Roy ([5], Proposition 4.1).

Proposition 2.5 *Let A be a ring and let I be an ideal of A . Let P be a projective A -module. Then any transvection of P/IP can be lifted to an automorphism of P .*

Definition 2.6 For a ring A , we say that projective stable range of A is $\leq r$ (notation: $\text{psr}(A) \leq r$) if for all projective A -modules P of rank $\geq r$ and $(a, p) \in \text{Um}(A \oplus P)$, we can find $q \in P$ such that $p + aq \in \text{Um}(P)$. Similarly, A has stable range $\leq r$ (notation: $\text{sr}(A) \leq r$) is defined the same way as $\text{psr}(A)$ but with P required to be free.

The following two results are due to Suslin, Vaserstein ([23], Corollary 17.3) and Mohan Kumar, Murthy, Roy ([11], Theorem 3.7) respectively.

Theorem 2.7 *Let $k \subset \overline{\mathbb{F}}_p$ be a field and let A be an affine k -algebra of dimension d . Then $\text{sr}(A) \leq \max(2, d)$. In particular, A^d is cancellative.*

Theorem 2.8 *Let A be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$. Suppose that A is regular when $d = 2$. Then $\text{psr}(A) \leq d$.*

The following result is due to Quillen [15] and Suslin [22].

Theorem 2.9 *Let A be a ring and let P be a projective $A[T]$ -module. Assume that P_f is free for some monic polynomial $f \in A[T]$. Then P is free.*

The following result is due to Wiemers ([25], Theorem 3.2, Corollary 3.4).

Proposition 2.10 *Let R be a ring of dimension d and $A = R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Let P be a projective A -module of rank $\geq \max(2, d + 1)$. Then*

- (i) $EL^1(A \oplus P, Y_m - 1)$ acts transitively on $\text{Um}^1(A \oplus P, Y_m - 1)$.
- (ii) the natural map $\text{Aut}_A(P) \rightarrow \text{Aut}_{A/(Y_m - 1)}(P/(Y_m - 1)P)$ is surjective.

We state two results due to Wiemers ([25], Lemma 4.2 and Theorem 4.3) respectively which are very crucial for our results.

Proposition 2.11 *Let A be a ring and let P be an A -module (need not be projective). Assume that there exists $p = [p_1, \dots, p_n] \in \text{Hom}_A(A^n, P)$, $\phi = [\phi_1, \dots, \phi_n]^t \in \text{Hom}_A(P, A^n)$ and $s_1, \dots, s_n \in A$ such that*

- (i) $(0 : s_i) = (0 : s_i^2)$ for $i = 1, \dots, n$,
- (ii) $(\phi_i(p_j))_{n \times n} = \text{diagonal}(s_1, \dots, s_n) = N$.

Let \mathcal{M} be the subgroup of $\text{GL}_n(A)$ consisting of all matrices $1_n + T.N^2$ for some matrix T . Then the map $\Phi : \mathcal{M} \rightarrow \text{Aut}_A(P)$, defined by $\Phi(1_n + T.N^2) = \text{Id}_P + p.T.N.\phi$ is a group homomorphism.

Theorem 2.12 *Let R be a ring of dimension d with $1/d! \in R$ and $A = R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Let P be a projective A -module of rank d . If $P/(X_1, \dots, X_n)P$ is cancellative, then P is cancellative. In particular, if projective R -modules of rank d are cancellative, then projective $R[X_1, \dots, X_n]$ -modules of rank d are also cancellative.*

We end this section by stating two results due to Keshari ([7], Theorem 3.5 and Theorem 4.4) and ([6], Theorem 3.10).

Theorem 2.13 *Let R be an affine algebra of dimension $n \geq 3$ over an algebraically closed field k with $1/(n-1)! \in k$. Let g, f_1, \dots, f_r be a R -regular sequence and $A = R[f_1/g, \dots, f_r/g]$. Let P' be a projective A -module of rank $n-1$ which is extended from R . Let $(a, p) \in \text{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then P is extended from R .*

Theorem 2.14 *Let R be an affine k -algebra of dimension $n \geq 3$, where k is a real closed field. Let $f \in R$ be an element not belonging to any real maximal ideal of A . Assume that either*

- (i) $A = R[f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a regular sequence in R or
- (ii) $A = R_f$.

Let P' be a projective A -module of rank $\geq n-1$ which is extended from R . Let $(a, p) \in \text{Um}(A \oplus P')$ and $P = A \oplus P'/(a, p)A$. Then P is extended from R .

3 Main Theorem

We begin this section with the following result which is very crucial for later use and seems to be well known to experts. Since we are unable to find an appropriate reference, we give the complete proof.

Proposition 3.1 *Let A be a ring of dimension d and let J be an ideal of A . Consider the cartesian square*

$$\begin{array}{ccc} C & \xrightarrow{i_1} & A \\ i_2 \downarrow & & \downarrow j_1 \\ A & \xrightarrow{j_2} & A/J \end{array}$$

Then C is finitely generated algebra over A of dimension d . In particular, if A is an affine algebra over a field k , then C is also an affine algebra over k .

Proof Recall that C is the subalgebra of $A \times A$ consisting of all elements (a, b) such that $a - b \in J$. First we will show that $C \xrightarrow{\sim} A \oplus J$, where $A \oplus J$ has the obvious ring structure, i.e. $(a, x) + (a', x') = (a + a', x + x')$ and $(a, x).(a', x') = (aa', ax' + a'x + xx')$ for $(a, x), (a', x') \in A \oplus J$.

We define $i_1 : A \oplus J \rightarrow A$ by $i_1(a, x) = a + x$ and $i_2 : A \oplus J \rightarrow A$ by $i_2(a, x) = a$. Then $j_1 i_1 = j_2 i_2$. It is enough to show that $A \oplus J$ satisfies the universal property of cartesian square. Let B be a ring and let $f_i : B \rightarrow A$ be ring homomorphism, $i = 1, 2$ such that $j_1 f_1 = j_2 f_2$. To show that there exists a unique ring homomorphism $F : B \rightarrow A \oplus J$ such that $i_1 F = f_1$ and $i_2 F = f_2$.

Define $F(b) = (f_2(b), f_1(b) - f_2(b))$. Since $j_1 f_1 = j_2 f_2$, $F : B \rightarrow A \oplus J$. Also it is clear that $i_1 F = f_1$ and $i_2 F = f_2$. It remains to show that F is a ring homomorphism. Clearly, $F(b + b') = F(b) + F(b')$ for $b, b' \in B$. We have

$$\begin{aligned} F(b).F(b') &= (f_2(b), f_1(b) - f_2(b)).(f_2(b'), f_1(b') - f_2(b')) \\ &= (f_2(b)f_2(b'), f_2(b)(f_1(b') - f_2(b')) + f_2(b')(f_1(b) - f_2(b)) + (f_1(b) - f_2(b))(f_1(b') - f_2(b'))) \\ &= (f_2(bb'), f_1(bb') - f_2(bb')) = F(bb'). \end{aligned}$$

Uniqueness of F follows from the fact that $i_1 F = f_1$ and $i_2 F = f_2$. This proves that $C \xrightarrow{\sim} A \oplus J$. If $J = (a_1, \dots, a_r)$, then $A \oplus J$ is generated by $(0, a_1), \dots, (0, a_r)$ over $A \oplus 0$, since if $x = a_1 x_1 + \dots + a_r x_r \in J$, then $(0, x) = (x_1, 0).(0, a_1) + \dots + (x_r, 0).(0, a_r)$. Hence $A \oplus J$ is a finitely generated algebra over A .

To show that $\dim A \oplus J = \dim A$, we show that $A \oplus J$ is integral over A . It is enough to show that $(0, a_i)$, $i = 1, \dots, r$ are integral over A . Clearly $(0, a_i)^2 - (a_i, 0)(0, a_i) = (0, 0)$. This proves the result. \square

Corollary 3.2 *Let A be a ring and let $s \in A$. Then the cartesian square of (A, A) over A/sA is $A[X]/(X^2 - sX)$.*

The following result is very crucial for later use.

Lemma 3.3 *Let A be a ring and let P be a projective A -module of rank r . Choose $s \in A$ satisfying the properties of (2.4). Assume that R^r is cancellative, where $R = A[X]/(X^2 - s^2 X)$. Then $\text{Aut}(A \oplus P, sA)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$.*

Proof Without loss of generality, we can assume that A is reduced. By (2.4), there exist $p_1, \dots, p_r \in P$ and $\phi_1, \dots, \phi_r \in \text{Hom}(P, A)$ such that P_s is free, $(\phi_i(p_j)) = \text{diagonal}(s, \dots, s)$, $sP \subset p_1 A + \dots + p_r A$ and s is a non-zero-divisor.

Let $(f, q) \in \text{Um}^1(A \oplus P, s^2 A)$. Since $f \in 1 + s^2 A$, by adding some multiple of f to q , we may assume that $q \in s^3 P$. Since $sP \subset p_1 A + \dots + p_r A$, we can write $q = f_1 p_1 + \dots + f_r p_r$ for some $f_i \in s^2 A$, $i = 1, \dots, r$. Note that $(f, f_1, \dots, f_r) \in \text{Um}_{r+1}(A, s^2 A)$.

By (3.2), R is the cartesian square of (A, A) over $A/s^2 A$.

$$\begin{array}{ccc} R & \xrightarrow{i_1} & A \\ i_2 \downarrow & & \downarrow j_1 \\ A & \xrightarrow{j_2} & A/(s^2) \end{array}$$

Patching unimodular rows (f, f_1, \dots, f_r) and $(1, 0, \dots, 0)$ over A/s^2A , we get a unimodular row $(c_0, c_1, \dots, c_r) \in \text{Um}_{r+1}(R)$. Since R^r is cancellative, there exists $\Theta \in \text{GL}_{r+1}(R)$ such that $(c_0, c_1, \dots, c_r)\Theta = (1, 0, \dots, 0)$. The projections of this equation gives

$$(f, f_1, \dots, f_r)\Psi = (1, 0, \dots, 0), \quad (1, 0, \dots, 0)\tilde{\Psi} = (1, 0, \dots, 0)$$

for certain matrices $\Psi, \tilde{\Psi} \in \text{GL}_{r+1}(A)$ such that $\Psi = \tilde{\Psi}$ modulo (s^2) . Hence $(f, f_1, \dots, f_r)\Psi\tilde{\Psi}^{-1} = (1, 0, \dots, 0)$, where $\Psi\tilde{\Psi}^{-1} = \Delta \in \text{GL}_{r+1}(A, s^2A)$.

Let $\Delta = 1 + TN^2$, where T is some matrix and $N = \text{diagonal}(1, s, \dots, s)$. Applying (2.11) with $n = r + 1$ and $(s_1, \dots, s_n) = (1, s, \dots, s)$, we get $\Phi(\Delta) = Id + pTN\phi \in \text{Aut}(A \oplus P, sA)$, where $p = [p_1, \dots, p_n] \in \text{Hom}(A^n, P)$ and $\phi = [\phi_1, \dots, \phi_n]^t \in \text{Hom}(P, A^n)$ with $(\phi_i(p_j)) = N = \text{diagonal}(1, s, \dots, s)$. We have

$$\begin{aligned} \Phi(\Delta)(f, f_1p_1 + \dots + f_rp_r) &= (Id + pTN\phi)(f, f_1p_1 + \dots + f_rp_r) \\ &= (f, f_1p_1 + \dots + f_rp_r) + pTN(f, f_1s, \dots, f_rs)^t \\ &= p(f, f_1, \dots, f_r)^t + pT(f_0, f_1s^2, \dots, f_rs^2)^t \\ &= p(1 + TN^2)(f, f_1, \dots, f_r)^t = p(1, 0, \dots, 0)^t = (1, 0). \end{aligned}$$

This proves the result. \square

Corollary 3.4 *Let A be a ring of dimension d and let P be a projective A -module of rank d . Choose $s \in A$ satisfying the properties of (2.4). Assume that R^r is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Proof We may assume that A is reduced. By (2.2), $A \oplus P$ is cancellative, hence, we need to show that $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$. Let $(f, q) \in \text{Um}(A \oplus P)$. Since s is a non-zero-divisor, $\dim A/s^2 < \dim A$. Hence, by (2.2), there exists $\theta \in E(\overline{A \oplus P})$ such that $\theta(\overline{f}, \overline{q}) = (1, 0)$, where “bar” denotes reduction modulo (s^2) . By (2.5), θ can be lifted to $\Theta \in \text{Aut}(A \oplus P)$ and $\Theta(f, q) \in \text{Um}^1(A \oplus P, s^2A)$. By (3.3), there exists $\Theta_1 \in \text{Aut}(A \oplus P)$ such that $\Theta_1\Theta(f, q) = (1, 0)$. This proves the result. \square

As a consequence of above result, we prove our first main result.

Theorem 3.5 *Let A be a ring of dimension d . Assume that for every finite extension R of A , R^d is cancellative. Then every projective A -module of rank d is cancellative.*

Proof Let P be a projective A -module of rank d . Choose $s \in A$ satisfying the properties of (2.4). If $R = A[X]/(X^2 - s^2X)$, then R is finite extension of A and hence R^d is cancellative. By (3.4), P is cancellative. \square

Lemma 3.6 *Let A be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$. Let P be a projective A -module of rank $d - 1$. Choose $s \in A$ satisfying the properties of (2.4). Assume that R^{d-1} is cancellative, where $R = A[X]/(X^2 - s^2X)$. Then P is cancellative.*

Proof We can assume that A is reduced and hence s is a non-zero-divisor. Since, by Suslin's result (1.1), every projective A -module of rank d is cancellative, it is enough to show that $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Let $(a, p) \in \text{Um}(A \oplus P)$. Let “bar” denotes reduction modulo s^2A . Then $\dim \bar{A} = d - 1 \geq 3$. By (2.8), there exists $q \in P$ such that $\bar{p} + \bar{a}q \in \text{Um}(\bar{P})$. Hence there exists $\bar{\sigma} \in E(\bar{A} \oplus \bar{P})$ such that $\bar{\sigma}(\bar{a}, \bar{p}) = (1, 0)$. Lifting $\bar{\sigma}$ to $\sigma \in \text{Aut}(A \oplus P)$ and replacing (a, p) by $\sigma(a, p)$, we may assume that $(a, p) \in \text{Um}^1(A \oplus P, s^2A)$. By (3.3), there exists $\Delta \in \text{Aut}(A \oplus P)$ such that $\Delta(a, p) = (1, 0)$. This proves the result. \square

Theorem 3.7 *Let A be an affine algebra of dimension $d \geq 4$ over $\bar{\mathbb{F}}_p$. Assume that if R is a finite extension of A , then R^{d-1} is cancellative. Then every projective A -module of rank $d-1$ is cancellative.*

Proof Let P be a projective A -module of rank $d-1$. Choose $s \in A$ satisfying the properties of (2.4). Since $R = A[X]/(X^2 - s^2X)$ is a finite extension of A , R^{d-1} is cancellative, by hypothesis. Applying (3.6), P is cancellative. \square

Proposition 3.8 *Let A be a ring and let P be a projective A -module of rank r . Choose $s \in A$ satisfying the properties of (2.4). If $\text{GL}_{r+1}(A, s^2A)$ acts transitively on $\text{Um}_{r+1}(A, s^2A)$, then $\text{Aut}(A \oplus P, sA)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$.*

Proof Let $(a, p) \in \text{Um}^1(A \oplus P, s^2A)$. Since $a = 1$ modulo s^2A , adding some multiple of a to p , we may assume that $p \in s^3P$. Since, by (2.4), $sP \subset p_1A + \dots + p_rA$, we get $p = a_1p_1 + \dots + a_rp_r$ for some $a_i \in s^2A$, $i = 1, \dots, r$. Note that $(a, a_1, \dots, a_r) \in \text{Um}_{r+1}(A, s^2A)$. By assumption, there exists $\Delta \in \text{GL}_{r+1}(A, s^2A)$ such that $\Delta(a, a_1, \dots, a_r) = (1, 0, \dots, 0)$.

Let $\Delta = 1 + TN^2$, where T is some matrix and $N = \text{diagonal}(1, s, \dots, s)$. Applying (2.11) with $n = r + 1$ and $(s_1, \dots, s_n) = (1, s, \dots, s)$, we get $\Phi(\Delta) = Id + pTN\phi \in \text{Aut}(A \oplus P)$. It is easy to see that $\Phi(\Delta) \in \text{Aut}(A \oplus P, sA)$. Further, as in the proof of (3.3), we can see that $\Phi(\Delta)(a, p) = (1, 0)$. This proves the result. \square

The following result generalizes ([23], Corollary 17.3).

Theorem 3.9 *Let $k \subset \bar{\mathbb{F}}_p$ be a field and let A be an affine algebra over k of dimension $d \geq 2$. Then every projective A -module of rank d is cancellative.*

Proof By Suslin-Vaserstein result ([23], Corollary 17.3), $\text{sr}(A) \leq d$. Hence every stably free A -module of rank d is free, i.e. A^d is cancellative. If B is a finite extension of A , then B is also affine k -algebra of dimension d and hence B^d is also cancellative. By (3.5), the result follows. \square

Remark 3.10 Let k be a field and let A be an affine k -algebra of dimension d . Assume that characteristic of k is either 0 or $p > d$. Further assume that $\text{cd}(k) \leq 1$, where “cd” stands for cohomological dimension [18]. Then A^d is cancellative (Suslin's result). The proof of this result is contained in [19] (see [12], 2.1 - 2.4).

In particular, if A is an affine k -algebra of dimension d , where k is a C_1 -field of characteristic 0 or $p > d$. Then A^d is cancellative. Note that we do not need k to be perfect in (1.1 (ii)). By (3.4), we get Bhatwadekar's result (1.1(ii)) that every projective A -module of rank d is cancellative.

4 Over algebraically closed fields

In this section, k will denote an algebraically closed field.

Proposition 4.1 *Let R be an affine k -algebra of dimension d with $1/d! \in k$. Let $f(T) \in R[T]$ be a monic polynomial. Assume that either*

(i) $A = R[T, 1/f(T)]$ or

(ii) $A = R[T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a regular sequence in $R[T]$.

Then A^d is cancellative.

Proof (i) Assume that $A = R[T, 1/f(T)]$ and let P be a stably free A -module of rank d . Since $A_{1+fk[f]}$ is an affine domain of dimension d over a C_1 -field $k(f)$, by Suslin's result (3.10), $P \otimes A_{1+fk[f]}$ is free. Hence, there exists $h \in 1 + fk[f]$ such that P_h is free. By ([12], Lemma 2.9), patching P and $(R[T]_h)^d$, we get a projective $R[T]$ -module Q of rank d such that $Q_f \xrightarrow{\sim} P$ and Q_h is free. Since $h \in R[T]$ is a monic polynomial, by (2.9), Q is free and hence P is free. This proves that A^d is cancellative.

(ii) Assume that $A = R[T, f_1/f, \dots, f_r/f]$ and let P be a stably free A -module of rank d . By (2.13), there exists a projective $R[T]$ -module Q of rank d such that $P \xrightarrow{\sim} Q \otimes A$. Since $P \oplus A \xrightarrow{\sim} (Q \otimes A) \oplus A$ is free, hence $(Q \oplus R[T]) \otimes R[T, 1/f]$ is free. Since f is a monic polynomial, by (2.9), $Q \oplus R[T]$ is free. By (2.12), $R[T]^d$ is cancellative. Hence Q is free and therefore P is free. This proves that A^d is cancellative. \square

Lemma 4.2 *Let R be a reduced ring of dimension d and $A = R[T, 1/f(T)]$ for some $f(T) \in R[T]$. Let P be a projective A -module. Then there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.4).*

Proof Let S be the set of non-zero-divisors of R . Then $S^{-1}R$ is a direct product of fields. Since $K[T, 1/g(T)]$ is a PID for any field K and $g(T) \in K[T]$, every projective $K[T, 1/g(T)]$ -module is free. Hence every projective module of constant rank over $S^{-1}R[T, 1/f(T)]$ is free. Now, it is easy to see that we can choose $s \in S$ satisfying the properties of (2.4). \square

Theorem 4.3 *Let R be a reduced affine k -algebra of dimension d with $1/d! \in k$. Let $f(T) \in R[T]$ be a monic polynomial and let $A = R[T, 1/f(T)]$. Let P be a projective A -module of rank d . By (4.2), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Then $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$.*

Proof Let $C = A[X]/(X^2 - s^2X) = B[T, 1/f(T)]$, where $B = R[X]/(X^2 - s^2X)$ is an affine k -algebra of dimension d . By (4.1), C^d is cancellative. Applying (3.3), we get that $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. \square

Remark 4.4 In (4.3), if every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$, then P will be cancellative. The same remark is applicable for (4.7, 5.6 and 5.8).

Lemma 4.5 *Let R be a reduced ring and $A = R[T, f_1/f, \dots, f_r/f]$ for some $f, f_1, \dots, f_r \in R[T]$. Let P be a projective A -module with trivial determinant. Then there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.4).*

Proof Let S be the set of non-zero-divisors of R . Then $S^{-1}R$ is a direct product of fields and $\dim S^{-1}A = 1$. As determinant of P is trivial, by (2.1), $S^{-1}P$ is free. Now, we can choose $s \in S$ satisfying the properties of (2.4). \square

Remark 4.6 Let $A = K[T, f(T)/g(T)]$, K is a field. We can assume that f and g have no common factors. Hence $(f, g) = K[T]$. Since $A_f = K[T, (fg)^{-1}]$ and $A_g = K[T, g^{-1}]$ are PID, A is a Dedekind domain. We do not know if all projective A -modules are free.

Theorem 4.7 *Let R be a reduced affine k -algebra of dimension d with $1/d! \in k$. Let $f(T) \in R[T]$ be a monic polynomial and $A = R[T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a regular sequence in $R[T]$. Let P be a projective A -module of rank d with trivial determinant. By (4.5), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Then $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$.*

Proof Let $C = A[X]/(X^2 - s^2X) = B[T, f_1/f, \dots, f_r/f]$, where $B = R[X]/(X^2 - s^2X)$ is an affine k -algebra of dimension d . Since B is a free R -module, f, f_1, \dots, f_r is a $B[T]$ -regular sequence. By (4.1), C^d is cancellative. Applying (3.3), we get that $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. \square

Theorem 4.8 *Let R be an affine $\overline{\mathbb{F}}_p$ -algebra of dimension $d \geq 3$, where $p > d$. Let $f(T) \in R[T]$ be a monic polynomial and $A = R[T, f_1/f, \dots, f_r/f]$ for some $f, f_1, \dots, f_r \in R[T]$. Then every projective A -module of rank d with trivial determinant is cancellative.*

Proof First we prove that A^d is cancellative. Let P be a stably free A -module of rank d . By Suslin's result (1.1(i)), we may assume that $P \oplus A$ is free. By ([7], Theorem 3.6), P is extended from $R[T]$. Now, we can complete the proof as in (4.1(ii)).

Let P be a projective A -module of rank d with trivial determinant. We may assume that A is reduced. By (4.5), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). If $C = A[Y]/(Y^2 - s^2Y)$, then, as in the previous paragraph, C^d is cancellative. By (3.3), $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. Applying (2.8) and (2.5), it is easy to see that every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2A)$ by an automorphism of $A \oplus P$. This proves that P is cancellative. \square

As a consequence of (4.8), we get the following result which extends a result of Murthy ([12], Corollary 2.13), where it is proved that A^d is cancellative.

Theorem 4.9 *Let $R = \overline{\mathbb{F}}_p[X_1, \dots, X_{d+1}]$ and let A be a subring of the fraction field of R with $R \subset A$. Suppose $p > d \geq 3$. Then all projective A -modules of rank d with trivial determinant are cancellative.*

Using (4.1) and following the proofs of ([12], Proposition 3.1 and Theorem 3.6), we get the following two results.

Corollary 4.10 *Let $1/(d-1)! \in k$ and $A = k[x_0, x_1, \dots, x_d]$, where $x_0^2 + x_1^2 + f(x_2, \dots, x_d) = 0$ for some $f \in k[x_2, \dots, x_d]$. Then A^{d-1} is cancellative*

Corollary 4.11 *Let $1/(d-1)! \in k$ and $A = k[x, y, t_1, \dots, t_{d-1}]$, where $x + x^s g(x, t_1, \dots, t_{d-1}) + x^r y + f(t_1, \dots, t_{d-1}) = 0$, with $d \geq 3$, $s \geq 2$, $r \geq 2$. Then A is a smooth d dimensional affine k -algebra. Further A^{d-1} is cancellative.*

5 Over real closed fields

In this section, k will denote a real closed field.

Proposition 5.1 *Let R be an affine k -algebra of dimension $d-1 \geq 2$ and let $f(T) \in R[T]$ be a monic polynomial. Assume that $f(T)$ does not belong to any real maximal ideal of $R[T]$ and either*

(i) $A = R[T, 1/f(T)]$ or

(ii) $A = R[T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a regular sequence in $R[T]$.

Then A^d is cancellative.

Proof (i) Let $A = R[T, 1/f(T)]$ and let P be a stably free A -module of rank d . Then $P \oplus A$ is free, by (2.2). By ([13], Theorem), P is extended from $R[T]$. Let Q be a projective $R[T]$ -module such that $P = Q \otimes A$. Then $(Q \oplus R[T])_f$ is free and f is a monic polynomial, hence $Q \oplus R[T]$ is free, by (2.9). By Plumstead's result [14], every projective $R[T]$ -module of rank $> \dim R$ is cancellative. Hence Q is free and therefore P is free.

(ii) Let $A = R[T, f_1/f, \dots, f_r/f]$ and let P be a stably free A -module of rank d . Then $P \oplus A$ is free, by (2.2). By (2.14), P is extended from $R[T]$. Let Q be a projective $R[T]$ -module of rank d such that $P \xrightarrow{\sim} Q \otimes A$. Since $(Q \otimes A) \oplus A$ is free, $(Q \oplus R[T]) \otimes R[T, 1/f]$ is free. As f is a monic polynomial, by (2.9), $Q \oplus R[T]$ is free. By Plumstead's result [14], Q is cancellative. Hence Q is free and so P is free. \square

Theorem 5.2 *Let R be an affine k -algebra of dimension $d-1 \geq 2$ and let $f(T) \in R[T]$ be a monic polynomial. Assume that $f(T)$ does not belong to any real maximal ideal of $R[T]$ and $A = R[T, 1/f(T)]$. Then every projective A -module of rank d is cancellative.*

Proof We may assume that R is reduced. Let P be a projective A -module of rank d . By (4.2), we can choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Let $C = A[X]/(X^2 - s^2X) = B[T, 1/f(T)]$, where $B = R[X]/(X^2 - s^2X)$. Since $B[T]$ is a finite extension of $R[T]$, any maximal ideal of $B[T]$ will contract to a maximal ideal of $R[T]$. Therefore, $f(T)$ does not belong to any real maximal ideal of $B[T]$. By (5.1), C^d is cancellative. Hence, by (3.4), P is cancellative. \square

Theorem 5.3 *Let R be an affine k -algebra of dimension $d - 1 \geq 2$ and let $f(T) \in R[T]$ be a monic polynomial. Assume that $f(T)$ does not belong to any real maximal ideal of $R[T]$ and $A = R[T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a regular sequence in $R[T]$. Then every projective A -module of rank d with trivial determinant is cancellative.*

Proof We may assume that R is reduced. Let P be a projective A -module of rank d with trivial determinant. By (4.5), we can choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Let $C = A[X]/(X^2 - s^2X) = B[T, f_1/f, \dots, f_r/f]$, where $B = R[X]/(X^2 - s^2X)$. Since $B[T]$ is a finite extension of $R[T]$, any maximal ideal of $B[T]$ will contract to a maximal ideal of $R[T]$. Therefore, $f(T)$ does not belong to any real maximal ideal of $B[T]$. Also, since $B[T]$ is a free $R[T]$ -module, f, f_1, \dots, f_r is a regular sequence in $B[T]$. By (5.1), C^d is cancellative. Hence, by (3.4), P is cancellative. \square

Proposition 5.4 *Let R be an affine k -algebra of dimension $d - 2 \geq 1$. Let $A = R[X, T, 1/f]$, where $f \in R[X, T]$ is a monic polynomial in T and f does not belong to any real maximal ideal of $R[X, T]$. Then A^{d-1} is cancellative.*

Proof Let P be a stably free A -module of rank $d - 1$. By (5.1), we may assume that $P \oplus A$ is free. By (2.14), P is extended from $R[X, T]$. Let Q be a projective $R[X, T]$ -module such that $P \xrightarrow{\sim} Q \otimes A$. Since $(Q \oplus R[X, T]) \otimes A$ is free and f is a monic polynomial, by (2.9), $Q \oplus R[X, T]$ is free. By Ravi Rao's result ([16], Theorem 2.5), every projective $R[X_1, \dots, X_n]$ -module of rank $> \dim R$ is cancellative. Hence Q is free and therefore P is free. \square

Corollary 5.5 *Let $A = k[X_1, \dots, X_d, 1/f]$. Assume that f does not belong to any real maximal ideal of $k[X_1, \dots, X_d]$. Then every projective A -module of rank $\geq d - 1$ is free.*

Proof Since $K_0(A) = \mathbb{Z}$, every projective A -module is stably free. Now the result follows from (5.4). \square

Theorem 5.6 *Let R be an affine k -algebra of dimension $d - 2 \geq 1$. Let $A = R[X, T, 1/f]$, where $f \in R[X, T]$ is a monic polynomial in T and f does not belong to any real maximal ideal of $R[X, T]$. Let P be a projective A -module of rank $d - 1$. Assume that there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Then $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$.*

Proof Let $C = A[Y]/(Y^2 - s^2Y) = B[X, T, 1/f]$, where $B = R[Y]/(Y^2 - s^2Y)$. Since $B[X, T]$ is a finite extension of $R[X, T]$, f does not belong to any real maximal ideal of $B[X, T]$. By (5.4), C^{d-1} is cancellative. By (3.3), $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2A)$. \square

Proposition 5.7 *Let R be an affine k -algebra of dimension $d-2 \geq 1$. Let $A = R[X, T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a $R[X, T]$ -regular sequence. Assume that f is a monic polynomial in T and f does not belong to any real maximal ideal of $R[X, T]$. Then A^{d-1} is cancellative.*

Proof Let P be a stably free A -module of rank $d-1$. By (5.1), $P \oplus A$ is free and hence by (2.14), P is extended from $R[X, T]$. Let Q be a projective $R[X, T]$ -module such that $P \xrightarrow{\sim} Q \otimes A$. Since $(Q \oplus R[X, T]) \otimes A$ is free, $(Q \oplus R[X, T]) \otimes R[X, T]_f$ is free. Since f is a monic polynomial, by (2.9), $Q \oplus R[X, T]$ is free. By Ravi Rao's result ([16], Theorem 2.5), every projective $R[X_1, \dots, X_n]$ -module of rank $> \dim R$ is cancellative. Hence Q is free and therefore P is free. \square

Theorem 5.8 *Let R be an affine k -algebra of dimension $d-2 \geq 1$. Let $A = R[X, T, f_1/f, \dots, f_r/f]$, where f, f_1, \dots, f_r is a $R[X, T]$ -regular sequence. Assume that f is a monic polynomial in T and f does not belong to any real maximal ideal of $R[X, T]$. Let P be a projective A -module of rank $d-1$. Assume that there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Then $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$.*

Proof Let $C = A[Y]/(Y^2 - s^2 Y) = B[X, T, f_1/f, \dots, f_r/f]$, where $B = R[Y]/(Y^2 - s^2 Y)$. Since B is a finite extension of R , every maximal ideal of $B[X, T]$ will contract to a maximal ideal of $R[X, T]$. Hence f does not belong to any real maximal ideal of $B[X, T]$. Also, as $B[X, T]$ is a free module over $R[X, T]$, f, f_1, \dots, f_r is a regular sequence in $B[X, T]$. By (5.7), C^{d-1} is cancellative. Hence, by (3.3), $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$. \square

6 Over Laurent polynomial rings

Lemma 6.1 *Let R be a reduced ring of dimension d and let $A = R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Let P be a projective A -module. Then there exists a non-zero-divisor $s \in R$ satisfying the properties of (2.4).*

Proof Let S be the set of non-zero-divisors of R . Then $S^{-1}R$ is a direct product of fields. Suslin ([21], Corollary 7.4) and Swan ([24], Theorem 1.1) independently proved that if K is a field or a PID, then every projective $K[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules are free. Hence $S^{-1}P$ is free. Now we can choose $s \in S$ satisfying the properties of (2.4). \square

Theorem 6.2 *Let R be a ring of dimension d and let $A = R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Let P be a projective A -module of rank $\geq d$. By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Assume that B^d is cancellative, where $B = A[T]/(T^2 - s^2 T)$. Then P is cancellative.*

Proof By (3.3), $\text{Aut}(A \oplus P)$ acts transitively on $\text{Um}^1(A \oplus P, s^2 A)$. Let “bar” denotes reduction modulo $s^2 A$. Since $\dim R/s^2 R < d$, by (2.3), $E(\overline{A \oplus P})$ acts transitively on $\text{Um}(\overline{A \oplus P})$. Further, by (2.5), every element of $E(\overline{A \oplus P})$ can be lifted to an element of $\text{Aut}(A \oplus P)$. Hence, every element of $\text{Um}(A \oplus P)$ can be taken to an element of $\text{Um}^1(A \oplus P, s^2 A)$ by an automorphism of $A \oplus P$. This proves that P is cancellative. \square

Theorem 6.3 *Let R be an affine algebra of dimension d over an algebraically closed field k with $1/d! \in k$. Let $A = R[X, X^{-1}]$. Then every projective A -module of rank d is cancellative.*

Proof We can assume that A is reduced. Let P be a projective A -module of rank d . By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Let $B = A[T]/(T^2 - s^2T) = B_1[X, X^{-1}]$, where $B_1 = R[T]/(T^2 - s^2T)$ is an affine algebra over k of dimension d . By (4.1), B^d is cancellative. Hence, applying (6.2), we get that P is cancellative. This proves the result. \square

Theorem 6.4 *Let R be a ring of dimension d . Let $A = R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ and let P be a projective A -module of rank $\geq d$. By (6.1), choose a non-zero-divisor $s \in R$ satisfying the properties of (2.4). Assume that B^d is cancellative, where $B = B_1[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ and $B_1 = R[T]/(T^2 - s^2T)$. Then the natural map $\text{Aut}(P) \rightarrow \text{Aut}_{A/(Y_m-1)}(P/(Y_m-1)P)$ is surjective.*

Proof When $\text{rank } P > d$, the result follows from (2.10). Hence, we assume that $\text{rank } P = d$. Let “bar” denotes reduction modulo $(Y_m - 1)A$. It is easy to see that we can assume that R is reduced.

Let $\tau \in \text{Aut}_{\bar{A}}(\bar{P})$, then by (2.10), we can lift $id_{\bar{A}} \oplus \tau \in \text{Aut}_{\bar{A}}(\bar{A} \oplus \bar{P})$ to an automorphism θ of $A \oplus P$. Let $\theta(1, 0) = (h, p) \in \text{Um}(A \oplus P, Y_m - 1)$. Assume that there exists $\mu \in \text{Aut}(A \oplus P, Y_m - 1)$ such that $\mu(h, p) = (1, 0)$. Then, we have the following commutative diagram

$$\begin{array}{ccccc} A \oplus P & \xrightarrow{\theta} & A \oplus P & \xrightarrow{\mu} & A \oplus P \\ \downarrow & & \downarrow & & \downarrow \\ \bar{A} \oplus \bar{P} & \xrightarrow{id_{\bar{A}} \oplus \sigma} & \bar{A} \oplus \bar{P} & \xrightarrow{id} & \bar{A} \oplus \bar{P} \end{array}$$

Note that $\Psi = \mu\theta \in \text{Aut}(A \oplus P)$ is a lift of $id_{\bar{A}} \oplus \tau$. Further $\Psi(1, 0) = (1, 0)$. Hence Ψ induces an automorphism $\Psi \in \text{Aut}(P)$ which is a lift of τ . Hence, it is enough to show that $\text{Aut}(A \oplus P, Y_m - 1)$ acts transitively on $\text{Um}(A \oplus P, Y_m - 1)$.

Let $(f, q) \in \text{Um}(A \oplus P, Y_m - 1)$. Let “tilde” denote reduction modulo s^3A . Since $\dim R/s^3R < \dim R$, by (2.10), $EL^1(\tilde{A} \oplus \tilde{P}, Y_m - 1)$ acts transitively on $\text{Um}^1(\tilde{A} \oplus \tilde{P}, Y_m - 1)$. After lifting the $EL^1(\tilde{A} \oplus \tilde{P}, Y_m - 1)$ transformations, we may assume that $(f, q) = (1, 0)$ modulo $s^3(Y_m - 1)A$.

Since $q \in s^3(Y_m - 1)P$, with the notation in (2.4), we can write $q = f_1p_1 + \dots + f_dp_d$ for some $f_i \in s^2(Y_m - 1)A$, $i = 1, \dots, d$. Note that $(f, f_1, \dots, f_d) \in \text{Um}_{d+1}(A, s^2(Y_m - 1))$.

Since B^d is cancellative, where B is the cartesian square of A, A over $A/(s^2)$, as in the proof of (3.3), there exists $\Delta \in \text{GL}_{d+1}(A, s^2A)$ such that $(f, f_1, \dots, f_d)\Delta = (1, 0, \dots, 0)$. Since $(f, f_1, \dots, f_d) \in \text{Um}_{d+1}(A, s^2(Y_m - 1))$, going modulo $(Y_m - 1)$, we get $(1, 0, \dots, 0)\Delta(Y_m - 1) = (1, 0, \dots, 0)$. Hence, if $\Theta = \Delta\Delta(Y_m - 1)^{-1}$, then $(f, f_1, \dots, f_d)\Theta = (1, 0, \dots, 0)$ and $\Theta \in \text{GL}_{d+1}(A, s^2(Y_m - 1))$.

Let $\Delta = 1 + TN^2$, where T is some matrix and $N = \text{diagonal}(1, s, \dots, s)$. Applying (2.11) with $n = d + 1$ and $(s_1, \dots, s_n) = (1, s, \dots, s)$, we get $\Psi = \Phi(\delta) \in \text{Aut}(A \oplus P, Y_m - 1)$ and $\Psi(f, f_1p_1 + \dots + f_dp_d) = (1, 0)$. This proves the result. \square

As an application of (6.3) and (6.4), we get the following result.

Corollary 6.5 *Let R be an affine algebra of dimension d over an algebraically closed field k with $1/d! \in k$. Let $A = R[X, X^{-1}]$ and let P be a projective A -module of rank d . Then the natural map $\text{Aut}(P) \rightarrow \text{Aut}_{A/(Y-1)}(P/(Y-1)P)$ is surjective.*

We end this section by stating four results which follow directly from (2.12) by applying (3.10), (6.3, 6.4), (1.1(iii)) and (3.9) respectively. Note that (6.7(i)) generalizes a result of Keshari ([8], Proposition A.9), where it is proved when A is a smooth affine algebra of dimension $d = 2$ and the determinant of P is trivial.

Theorem 6.6 *Let A be an affine algebra of dimension d over a C_1 -field k with $1/d! \in A$ and $R = A[X_1, \dots, X_n]$. Then every projective R -module of rank d is cancellative.*

Theorem 6.7 *Let A be an affine algebra of dimension d over an algebraically closed field k with $1/d! \in k$ and $R = A[X_1, \dots, X_n, Y^{\pm 1}]$. Let P be a projective R -module of rank d . Then*

- (i) *P is cancellative and*
- (ii) *the natural map $\text{Aut}(P) \rightarrow \text{Aut}(P/(Y-1)P)$ is surjective.*

Theorem 6.8 *Let A be a finitely generated algebra over \mathbb{Z} of dimension d with $1/d! \in A$. Then all projective $A[X_1, \dots, X_n]$ -modules of rank d are cancellative.*

Theorem 6.9 *Let $k \subset \overline{\mathbb{F}}_p$ be a field and let A be an affine k -algebra of dimension d . Assume that $p > d$. Then all projective $A[X_1, \dots, X_n]$ -modules of rank d are cancellative.*

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